Momentum Equations in Spherical Coordinates

- For a variety of reasons, it is useful to express the vector momentum equation for a rotating earth as a set of scalar component equations.
- The use of latitude-longitude coordinates to describe positions on earth’s surface makes it convenient to write the momentum equations in spherical coordinates.
- The coordinate axes are $(\lambda, \phi, z)$ where $\lambda$ is longitude, $\phi$ is latitude, and $z$ is height.

Orientation of Coordinate Axes

The x- and y-axes are customarily defined to point east and north, respectively, such that

$$dx = a \cos \phi \, d\lambda$$
and
$$dy = a \, d\phi$$

Thus the horizontal velocity components are

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}$$
A Complication of Spherical Coordinates

When the x and y coordinates are defined in this way, the coordinate system is not strictly Cartesian, because the directions of the unit vectors depend on their position on the earth’s surface.

This dependence on position can be accounted for mathematically (see Holton 2.3) by adding terms to each component of the total derivative:

\[
\frac{d\vec{V}}{dt} = \left( \frac{du}{dt} - \frac{uv \tan \phi + uw}{a} \right) \hat{i} + \left( \frac{dv}{dt} + \frac{u^2 \tan \phi + vw}{a} \right) \hat{j} + \left( \frac{dw}{dt} - \frac{u^2 + v^2}{a} \right) \hat{k}
\]

Vector momentum equation in rotating coordinates

\[
\frac{d\vec{V}}{dt} = -2\vec{\Omega} \times \vec{V} - \frac{1}{\rho} \nabla p + \vec{g} + \vec{F}_r
\]

Total derivative

\[
\frac{du}{dt} - \frac{uv \tan \phi + uw}{a} = \ldots
\]

\[
\frac{dv}{dt} + \frac{u^2 \tan \phi + vw}{a} = \ldots
\]

\[
\frac{dw}{dt} - \frac{u^2 + v^2}{a} = \ldots
\]
Vector momentum equation in rotating coordinates

\[ \frac{d\vec{V}}{dt} = -2\Omega \times \vec{V} - \frac{1}{\rho} \nabla p + \vec{g} + \vec{F}_r \]

Coriolis acceleration

\[-2\Omega \times \vec{v} = 2\begin{bmatrix}
\Omega \sin \phi & 0 & -\Omega \cos \phi \\
0 & \Omega & 0 \\
-\Omega \cos \phi & 0 & \Omega \sin \phi
\end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = (2 \Omega v \sin \phi - 2 \Omega w \cos \phi \dot{y} - 2 \Omega u \sin \phi \dot{x}) + 2 \Omega u \cos \phi \dot{z} \]

Pressure gradient term

\[ \nabla p = \frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k} \]

\[ \begin{align*}
\frac{du}{dt} - \frac{uv \tan \phi}{a} + \frac{uw}{a} &= 2\Omega v \sin \phi - 2\Omega w \cos \phi - \frac{1}{\rho} \frac{\partial p}{\partial x} + \ldots \\
\frac{dv}{dt} + \frac{u^2 \tan \phi}{a} + \frac{vw}{a} &= -2\Omega u \sin \phi - \frac{1}{\rho} \frac{\partial p}{\partial y} + \ldots \\
\frac{dw}{dt} - \frac{u^2 + v^2}{a} &= 2\Omega u \cos \phi - \frac{1}{\rho} \frac{\partial p}{\partial z} + \ldots 
\end{align*} \]
Vector momentum equation in rotating coordinates

\[ \frac{d\vec{V}}{dt} = -2\Omega \times \vec{V} - \frac{1}{\rho} \nabla \rho \vec{p} + \vec{g} + \vec{F}_r \]

Gravity

\[ \vec{g} = -g \hat{k} \quad g \text{ is a positive scalar } = 9.8 \text{ m s}^{-2} \text{ at earth’s surface} \]

Friction

\[ \vec{F}_r = F_{\alpha x}\hat{i} + F_{\alpha y}\hat{j} + F_{\alpha z}\hat{k} \]

\[ \frac{du}{dt} - \frac{uv \tan \phi}{a} + \frac{uw}{a} = 2\Omega v \sin \phi - 2\Omega w \cos \phi - \frac{1}{\rho} \frac{\partial \rho \vec{p}}{\partial x} + F_{\alpha x} \]

\[ \frac{dv}{dt} + \frac{u^2 \tan \phi}{a} + \frac{vw}{a} = -2\Omega u \sin \phi - \frac{1}{\rho} \frac{\partial \rho \vec{p}}{\partial y} + F_{\alpha y} \]

\[ \frac{dw}{dt} - \frac{u^2 + v^2}{a} = 2\Omega u \cos \phi - \frac{1}{\rho} \frac{\partial \rho \vec{p}}{\partial z} - g + F_{\alpha z} \]
Momentum Equations in Spherical Coordinates

\[
\begin{align*}
\frac{du}{dt} &= \frac{uv \tan \phi}{a} + \frac{uw}{a} - \frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi - 2\Omega w \cos \phi + F_{rx} \\
\frac{dv}{dt} &= \frac{u^2 \tan \phi}{a} + \frac{vw}{a} - \frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \phi + F_{ry} \\
\frac{dw}{dt} &= \frac{u^2 + v^2}{a} - \frac{1}{\rho} \frac{\partial p}{\partial z} + 2\Omega u \cos \phi - g + F_{rz}
\end{align*}
\]

Momentum Equations in Spherical Coordinates

\[
\begin{align*}
\frac{du}{dt} &= \frac{uv \tan \phi}{a} + \frac{uw}{a} - \frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi - 2\Omega w \cos \phi + F_{rx} \\
\frac{dv}{dt} &= \frac{u^2 \tan \phi}{a} + \frac{vw}{a} - \frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \phi + F_{ry} \\
\frac{dw}{dt} &= \frac{u^2 + v^2}{a} - \frac{1}{\rho} \frac{\partial p}{\partial z} + 2\Omega u \cos \phi - g + F_{rz}
\end{align*}
\]

- total derivative
- pressure gradient
- Coriolis
- gravity friction
Conservation of Mass

- There are three fundamental principles of physics that are important for fluid dynamics: conservation of momentum, conservation of mass, and conservation of energy.
- Conservation of mass means that mass cannot be created or destroyed.
- Thus we can derive an equation that accounts for the redistribution of mass in the atmosphere (or ocean).

\[
\partial V = \partial x \partial y \partial z
\]

We consider a very small volume element of air that is centered at the point \((x_0, y_0, z_0)\).

Conservation of mass requires that the local rate of change of mass must equal the net rate of mass inflow per unit volume.
Rate of inflow of mass through left wall: $M_{Lx}$

Rate of outflow of mass through right wall: $M_{Rx}$

Net inflow of mass into the volume (x-direction only): $M_x = M_{Lx} - M_{Rx}$

Method: Use Taylor expansion to develop mathematical expressions for the inflow of mass through the walls of this fluid element.

Taylor series expansion:

$f(x) = f(x_0) + f'(x_0)(x-x_0) + \text{higher order terms}$

In this case (neglecting higher order terms):

$\rho u(x) = \rho u(x_0) + \frac{\partial}{\partial x} (\rho u)(x-x_0)$

Therefore, we can express the rates of inflow and outflow per unit area as:

$M_{Lx} = \left( \rho u - \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \right)$

$M_{Rx} = \left( \rho u + \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \right)$
Since the area of the left and right faces of the volume is $\delta y \delta z$,

$$
\left[ \rho u - \frac{\partial}{\partial x} \left( \rho u \frac{\partial x}{2} \right) \delta y \delta z \right] - \left[ \rho u + \frac{\partial}{\partial x} \left( \rho u \frac{\partial x}{2} \right) \delta y \delta z \right] = - \frac{\partial}{\partial x} \left( \rho u \right) \delta x \delta y \delta z
$$

is the net rate of mass flow into the volume due to the $x$ velocity component.

We can derive similar expressions for the net rate of mass flow due to the $y$ and $z$ velocity components. Summing over all three components yields the net rate of mass inflow:

$$
- \left[ \frac{\partial}{\partial x} \left( \rho u \right) + \frac{\partial}{\partial y} \left( \rho v \right) + \frac{\partial}{\partial z} \left( \rho w \right) \right] \delta x \delta y \delta z = \text{net rate of mass inflow}
$$

$$
- \nabla \cdot \left( \rho \vec{v} \right) = \text{net rate of mass inflow per unit volume}
$$

The net rate of mass inflow per unit volume must equal the rate of mass increase per unit volume, which is the local rate of change of density.

$$
- \nabla \cdot \left( \rho \vec{v} \right) = \frac{\partial \rho}{\partial t}
$$

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
$$

**Mass divergence form of the continuity equation**
To derive an alternative form of the continuity equation, start with the mass divergence form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Now apply the vector identity:

$$\nabla \cdot (\rho \mathbf{v}) = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0$$

Next apply Euler's relationship:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

$$\frac{1}{\rho} \frac{d\rho}{dt} + \nabla \cdot \mathbf{v} = 0$$

Velocity divergence form of the continuity equation