Are All Of These Terms Important?

\[
\begin{align*}
\frac{du}{dt} - \frac{uv \tan \phi}{a} + \frac{uw}{a} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi - 2\Omega w \cos \phi + F_{rx} \\
\frac{dv}{dt} + \frac{u^2 \tan \phi}{a} + \frac{vw}{a} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \phi + F_{ry} \\
\frac{dw}{dt} - \frac{u^2 + v^2}{a} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + 2\Omega u \cos \phi - g + F_{rz}
\end{align*}
\]

Scale Analysis

- **Goal:** To determine relative importance of the terms in the basic equations for particular scales of motion.
- **Approach:** Estimate the following quantities
  1) The magnitude of the field variables.
  2) The amplitudes of fluctuations in the field variables. (Used to estimate derivatives.)
  3) The characteristic length, depth and time scales on which these fluctuations occur.
Scaling Quantities

$U = \text{horizontal velocity scale}$

$W = \text{vertical velocity scale}$

$L = \text{length scale}$

$H = \text{depth scale}$

$\delta P = \text{horizontal pressure fluctuation}$

$T = \text{time scale (advective)} = \frac{L}{U}$

$P_0 = \text{surface pressure scale}$

Values of Scaling Quantities
(midlatitude large-scale motions)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Atmosphere</th>
<th>Ocean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>$10 \text{ m s}^{-1}$</td>
<td>$10^{-1} \text{ m s}^{-1}$</td>
</tr>
<tr>
<td>$W$</td>
<td>$10^{-2} \text{ m s}^{-1}$</td>
<td>$10^{-4} \text{ m s}^{-1}$</td>
</tr>
<tr>
<td>$L$</td>
<td>$10^6 \text{ m}$</td>
<td>$10^6 \text{ m}$</td>
</tr>
<tr>
<td>$H$</td>
<td>$10^4 \text{ m}$</td>
<td>$10^3 \text{ m}$</td>
</tr>
<tr>
<td>$\delta P$ (horizontal)</td>
<td>$10^3 \text{ Pa}$</td>
<td>$10^4 \text{ Pa}$</td>
</tr>
<tr>
<td>$P_0$</td>
<td>$10^5 \text{ Pa}$</td>
<td>$10^7 \text{ Pa}$</td>
</tr>
<tr>
<td>$T$</td>
<td>$10^5 \text{ s}$</td>
<td>$10^7 \text{ s}$</td>
</tr>
</tbody>
</table>
Physical Constants

\[ g \approx 10 \text{ m s}^{-2} \quad \text{gravity} \]
\[ a \approx 10^7 \text{ m} \quad \text{radius of earth} \]
\[ \phi_0 = 45^\circ \]
\[ f_0 = 2\Omega \sin \phi_0 = 2\Omega \cos \phi_0 = 10^{-4} \text{ s}^{-1} \]
\[ \rho_a = 1 \text{ kg m}^{-3} \quad \text{density} \]
\[ \rho_o = 10^3 \text{ kg m}^{-3} \]
\[ v_a = 10^{-5} \text{ m}^2 \text{ s}^{-1} \quad \text{viscosity} \]
\[ v_o = 10^{-6} \text{ m}^2 \text{ s}^{-1} \]

Scaling The Horizontal Momentum Equations

\[
\begin{align*}
\frac{du}{dt} - \frac{u}{a} \tan \phi + \frac{uw}{a} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi - 2\Omega u \cos \phi + f_y \\
\frac{dv}{dt} + \frac{u}{a} \tan \phi + \frac{vw}{a} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \phi + f_x
\end{align*}
\]

\[
\begin{align*}
\frac{U^2}{L} &\quad \frac{U^2}{a} &\quad \frac{UW}{a} &\quad \frac{\delta P}{\rho L} &\quad f_0 U &\quad f_0 W &\quad \nu U \\
10^{-4} &\quad 10^{-5} &\quad 10^{-8} &\quad 10^{-3} &\quad 10^{-3} &\quad 10^{-6} &\quad 10^{-12}
\end{align*}
\]
Scaled Horizontal Momentum Equations
(a.k.a. Equations of Motion)

\[
\begin{align*}
\frac{du}{dt} &= -\frac{\partial p}{\partial x} + 2\Omega v \sin \phi \\
\frac{dv}{dt} &= -\frac{\partial p}{\partial y} - 2\Omega u \sin \phi
\end{align*}
\]

Scaling The Vertical Momentum Equation

\[
\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + 2\Omega \cos \phi - g + \tau
\]

| \[
\frac{UW}{L} \quad \frac{U^2}{a} \quad \frac{P_0}{\rho H} \quad f_0U \quad g \quad \frac{vW}{H^2}
\] |
<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>10^{-7}</td>
<td>10^6</td>
<td>10</td>
<td>10^{-3}</td>
<td>10</td>
<td>10^{-15}</td>
<td></td>
</tr>
</tbody>
</table>
The Hydrostatic Approximation

\[
\frac{\partial p}{\partial z} = -\rho g
\]

- For midlatitude synoptic-scale motions, vertical accelerations are very small compared to the vertical pressure gradient and gravity terms.
- This implies that vertical velocity cannot be determined from the vertical component of the momentum equation.

Geostrophic Balance

\[
\begin{align*}
\frac{du}{dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi \\
\frac{dv}{dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \phi
\end{align*}
\]

- There is an approximate balance between the pressure gradient and Coriolis terms.
- Retaining only these two terms leads to the geostrophic approximation.
- The geostrophic approximation is a diagnostic relationship that cannot be used to predict the evolution of the velocity field.
The Geostrophic Approximation

\[-fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad \text{Coriolis parameter} \]
\[fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} \]
\[f \equiv 2\Omega \sin \phi\]

The Geostrophic Wind

- The horizontal velocity field that satisfies the geostrophic approximation is known as the geostrophic wind.

\[v_g \equiv \frac{1}{\rho f} \frac{\partial p}{\partial x} \quad \vec{v}_g = \hat{k} \times \frac{1}{\rho f} \nabla p \]
\[u_g \equiv -\frac{1}{\rho f} \frac{\partial p}{\partial y} \]
Scale Analysis of Continuity Equation

When scaling the continuity equation, it is important to recognize that large variations in density are only relevant in the vertical. So we represent the density field as the sum of a horizontal reference value and a deviation from that value:

\[ \rho(x, y, z, t) = \rho_0(z) + \rho'(x, y, z, t) \]

Substitute into velocity divergence form of the continuity equation:

\[
\frac{1}{\rho} \frac{d\rho}{dt} + \nabla \cdot \vec{V} = 0
\]

\[
\frac{1}{\rho_0} \left[ \frac{\partial}{\partial t} (\rho_0 + \rho') + \vec{V} \cdot \nabla (\rho_0 + \rho') \right] + \nabla \cdot \vec{V} = 0
\]

\[
\frac{1}{\rho_0} \left[ \left( \frac{\partial \rho'}{\partial t} + \frac{\partial \rho'}{\partial z} \right) + u \left( \frac{\partial \rho'}{\partial x} + \frac{\partial \rho'}{\partial y} \right) + v \left( \frac{\partial \rho'}{\partial x} + \frac{\partial \rho'}{\partial y} \right) + w \left( \frac{\partial \rho_0}{\partial z} + \frac{\partial \rho'}{\partial z} \right) \right] + \nabla \cdot \vec{V} = 0
\]

If only the first three terms in the above equation were present, then the atmosphere would be incompressible.

The above result implies that the atmosphere behaves as if it were incompressible for purely horizontal motions (i.e., when \( w = 0 \)).

A: \( \frac{\rho' U}{\rho_0 L} = 10^{-7} \text{s}^{-1} \)

B: \( \frac{W}{H} = 10^{-6} \text{s}^{-1} \)

C: \( 10^{-3} \frac{U}{L}, \frac{W}{H} = 10^{-6} \text{s}^{-1} \)
Incompressible Fluids

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{V} = 0 \]

If a fluid is incompressible, its density is constant following the fluid motion. Thus the velocity divergence must be zero for an incompressible fluid.

In the atmosphere, the compressibility associated with the height dependence of density is important when there is vertical motion.

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + w \frac{d}{dz} (\ln \rho_0) = 0 \]

Incompressibility is a very good approximation in the ocean, even when vertical motions are present.