Conservation of Energy

- For a system in thermodynamic equilibrium, the first law of thermodynamics states that the change in internal energy of the system is equal to the difference between the heat added to the system and the work done by the system.
- The first law of thermodynamics applies to a moving fluid parcel.
- The rate of change of total thermodynamic energy is equal to the rate of diabatic heating plus the rate at which work is done on the parcel by external forces.

External Forces

- Work can be done on the fluid parcel by both surface forces and body forces.
- Surface force: pressure gradient.
- Body forces: gravity and Coriolis.
- The Coriolis force does no work because it is always perpendicular to motion of the fluid parcel. Force must be along the direction of motion for work to be done.
Thermodynamic Energy Equation

The derivation in section 2.6 of Holton yields a basic form of the thermodynamic energy equation:

\[ c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = Q \]

\( c_v \) is the specific heat at constant volume.

Rate of change of internal energy
Conversion between thermal and mechanical energy
Diabatic heating rate

Taking the total derivative of the equation of state:

\[ p \frac{d\alpha}{dt} + \alpha \frac{dp}{dt} = R \frac{dT}{dt} \]

then substituting,

\[ c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = Q \]

\( c_p = c_v + R \) is the specific heat at constant pressure.

\[ c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = Q \]

Expanding the total derivative of \( p \) and applying the hydrostatic approximation:

\[ c_p \frac{dT}{dt} = Q + \alpha \frac{dp}{dt} + \alpha \left( \vec{\nabla} \cdot \vec{p} \right) - gw \]

Scaling can be used to simplify this equation by noting that \( \alpha \frac{dp}{dt} \) and \( \alpha \left( \vec{\nabla} \cdot \vec{p} \right) \) are very small:

\[ \frac{dT}{dt} = \frac{Q}{c_p} - \frac{g}{c_p} w \]

Expanding the total derivative of \( T \) and substituting lapse rate definitions:

\[ \frac{\partial T}{\partial t} = \frac{Q}{c_p} - (\Gamma_d - \Gamma)w - \vec{V} \cdot \vec{\nabla} T \]

\( \Gamma_d = g/c_p, \quad \Gamma = -\partial T/\partial z \)
Temperature Tendency Equation

\[ \frac{\partial T}{\partial t} = \frac{Q}{c_p} - (\Gamma_d - \Gamma) w \cdot \nabla T \]

A: Diabatic Heating
1) Sensible heating
2) Latent heating (change of phase, i.e., evaporation, condensation)
3) Radiative heating

B: Adiabatic Effects
\( \Gamma_d - \Gamma \) is a measure of stability
Upward motion in a stable atmosphere is a cooling process.

C: Horizontal Advection Term
Often this is the largest term.

Entropy Form of Energy Equation

Starting with \( c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = Q \), dividing by \( T \), and applying
the equation of state yields the entropy form of the first law of thermodynamics:

\[ c_p \frac{d \ln T}{dt} - R \frac{d \ln p}{dt} = \frac{Q}{T} = \frac{ds}{dt} \]

This equation gives the rate of change of entropy per unit mass (s) following the fluid motion for a thermodynamically reversible process. A thermodynamically reversible process is one in which a system changes its thermodynamic state and then returns to its original state without changing the environment. Such a process, in which there are no sources or sinks of heat, is called an adiabatic process.
In the atmosphere Poisson’s equation can be applied:

\[ \theta = T \left( \frac{p_0}{p} \right)^{R/c_p} \]

where \( \theta \) is the potential temperature and \( p_0 \) is a standard pressure (usually 1000 hPa). Taking the natural logarithm of Poisson’s equation and applying it to the entropy form of the energy equation yields:

\[ \frac{ds}{dt} = c_p \frac{d \ln \theta}{dt} \]

This equation has important implications for atmospheric adiabatic processes. If the motion is adiabatic, then \( d\theta/dt = 0 \), which implies that \( \theta \) is conserved following the fluid motion. Thus \( \theta \) is a tracer of 3-D atmospheric trajectories, as long as there is no diabatic heating (e.g., clouds, convective processes).

**Primitive Equations**

\[
egin{align*}
\frac{du}{dt} - fv &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{dv}{dt} + fu &= -\frac{1}{\rho} \frac{\partial p}{\partial y} \\
\frac{dp}{dz} &= -\rho g \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= -\frac{1}{\rho} \frac{dp}{dt} \\
c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} &= Q \\
p &= \rho RT
\end{align*}
\]

6 equations with 6 dependent variables:

\( u, v, w, p, \rho, T \)
Primitive Equations

- These equations are called the primitive equations because they are very close to the original equations on which numerical weather prediction methods were first attempted.
- Modern numerical weather prediction models use the primitive equations almost exclusively.
- As listed on the previous slide, the primitive equations describe a dry atmosphere.
- A seventh equation, representing conservation of water vapor, is required to fully describe a moist atmosphere.