Continuity Equation in Pressure Coordinates

Here we will derive the continuity equation from the principle that mass is conserved for a parcel following the fluid motion (i.e., there is no flow across the boundaries of the parcel). This implies that

$$\delta M = \rho \delta V = \rho \delta x \delta y \delta z = -\frac{\delta x \delta y \delta p}{g}$$

is conserved following the fluid motion:

$$\frac{1}{\delta M} \frac{d(\delta M)}{dt} = 0$$

Taking the limit as $\delta x, \delta y, \delta p \to 0$,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Continuity equation in pressure coordinates
Determining Vertical Velocities

- Typical large-scale vertical motions in the atmosphere are of the order of 0.01-0.1 m/s.
- Such motions are very difficult, if not impossible, to measure directly. Typical observational errors for wind measurements are ~1 m/s.
- Quantitative estimates of vertical velocity must be inferred from quantities that can be directly measured with sufficient accuracy.

Vertical Velocity in P-Coordinates

The equivalent of the vertical velocity in p-coordinates is:

$$\omega = \frac{dp}{dt} = \frac{\partial p}{\partial t} + \vec{V} \cdot \nabla p + w \frac{\partial p}{\partial z}$$

Based on a scaling of the three terms on the r.h.s., the last term is at least an order of magnitude larger than the other two. Making the hydrostatic approximation yields

$$\omega \approx w \frac{\partial p}{\partial z} = -\rho g w$$

**Typical large-scale values:**
- for $w$, 0.01 m/s = 1 cm/s
- for $\omega$, 0.1 Pa/s = 1 μbar/s
The Kinematic Method

By integrating the continuity equation in \((x, y, p)\) coordinates, \(\omega\) can be obtained from the mean divergence in a layer:

\[
\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0 \quad \text{continuity equation in } (x, y, p) \text{ coordinates}
\]

\[
\int_{p_1}^{p_2} \partial \omega = -\int_{p_1}^{p_2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p \partial p \quad \text{rearrange and integrate over the layer}
\]

\[
\omega(p_2) - \omega(p_1) = (p_1 - p_2) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p \quad \text{overbar denotes pressure-weighted vertical average}
\]

To determine vertical motion at a pressure level \(p_2\), assume that \(p_1 = \text{surface pressure}\) and there is no vertical motion at the surface.

\[
\omega(p_2) - \omega(p_1) = (p_1 - p_2) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p \quad \text{No vertical motion at surface.}
\]

Estimating vertical motion at the top of a surface-based layer using the kinematic method:

\[
\omega_{\text{top}} \propto \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p
\]

If \(\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p < 0\), then \(\omega_{\text{top}} < 0\), which implies rising motion.

If \(\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p > 0\), then \(\omega_{\text{top}} > 0\), which implies sinking motion.
Vertical Profiles of Divergence and Vertical Motion

For well-developed cyclones and anticyclones in midlatitudes, the level of nondivergence is typically between 500 and 700 hPa.

Calculating Horizontal Divergence

To compute horizontal divergence at \((x_0, y_0)\), evaluate derivatives using centered finite differences:

\[
\begin{align*}
\left( \frac{\partial u}{\partial x} \right)_{x_0, y_0} & \approx \frac{u(x_0 + \Delta x, y_0) - u(x_0 - \Delta x, y_0)}{2\Delta x} \\
\left( \frac{\partial v}{\partial y} \right)_{x_0, y_0} & \approx \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0 - \Delta y)}{2\Delta y}
\end{align*}
\]
Example: \[ \Delta x = \Delta y = 50 \text{ km} \]

\[ u = 7.7 \text{ m/s} \quad u = 8.3 \text{ m/s} \]

\[ v = 3.5 \text{ m/s} \quad v = 4.0 \text{ m/s} \]

Divergence = \(1 \times 10^{-6} \text{ s}^{-1}\)

\[ \left( \frac{\partial u}{\partial x} \right)_{x_0, y_0} \approx \frac{8.3 \text{ m/s} - 7.7 \text{ m/s}}{2(50 \text{ km})} = \frac{0.6 \text{ m/s}}{10^5 \text{ m}} = 6 \times 10^{-6} \text{ s}^{-1} \]

\[ \left( \frac{\partial v}{\partial y} \right)_{x_0, y_0} \approx \frac{3.5 \text{ m/s} - 4.0 \text{ m/s}}{2(50 \text{ km})} = \frac{-0.5 \text{ m/s}}{10^5 \text{ m}} = -5 \times 10^{-6} \text{ s}^{-1} \]

\[ \nabla \cdot \vec{V} = 6 \times 10^{-6} \text{ s}^{-1} - 5 \times 10^{-6} \text{ s}^{-1} = 1 \times 10^{-6} \text{ s}^{-1} \]

Now, we introduce some small errors (0.1-0.2 m/s) in the wind estimates:

\[ \left( \frac{\partial u}{\partial x} \right)_{x_0, y_0} \approx \frac{8.1 \text{ m/s} - 7.6 \text{ m/s}}{2(50 \text{ km})} = \frac{0.5 \text{ m/s}}{10^5 \text{ m}} = 5 \times 10^{-6} \text{ s}^{-1} \]

\[ \left( \frac{\partial v}{\partial y} \right)_{x_0, y_0} \approx \frac{3.3 \text{ m/s} - 3.9 \text{ m/s}}{2(50 \text{ km})} = \frac{-0.6 \text{ m/s}}{10^5 \text{ m}} = -6 \times 10^{-6} \text{ s}^{-1} \]

\[ \nabla \cdot \vec{V} = 5 \times 10^{-6} \text{ s}^{-1} - 6 \times 10^{-6} \text{ s}^{-1} = -1 \times 10^{-6} \text{ s}^{-1} \]

Divergence estimates using the kinematic method are very sensitive to small errors.
Typical Values for Horizontal Divergence

- Atmosphere: $10^{-5}$ to $10^{-6}$ s$^{-1}$
- Ocean: $10^{-7}$ to $10^{-8}$ s$^{-1}$
- For large-scale weather systems in midlatitudes, the horizontal velocity is close to geostrophic balance. If not for the small contribution due to the variation of the Coriolis parameter with latitude, the geostrophic wind would be nondivergent.
- Horizontal divergence in the atmosphere arises primarily due to the small departures of the wind from geostrophic balance.

Divergence in Natural Coordinates

\[
\nabla_H \cdot \vec{V} = -V \frac{\partial \beta}{\partial n} + \frac{\partial V}{\partial s}
\]

$\beta$ = angle between streamline and flow direction

$\beta > 0$  $\beta < 0$

$-V \frac{\partial \beta}{\partial n} < 0$

"directional convergence"

$\frac{\partial V}{\partial s} < 0$

"speed convergence"