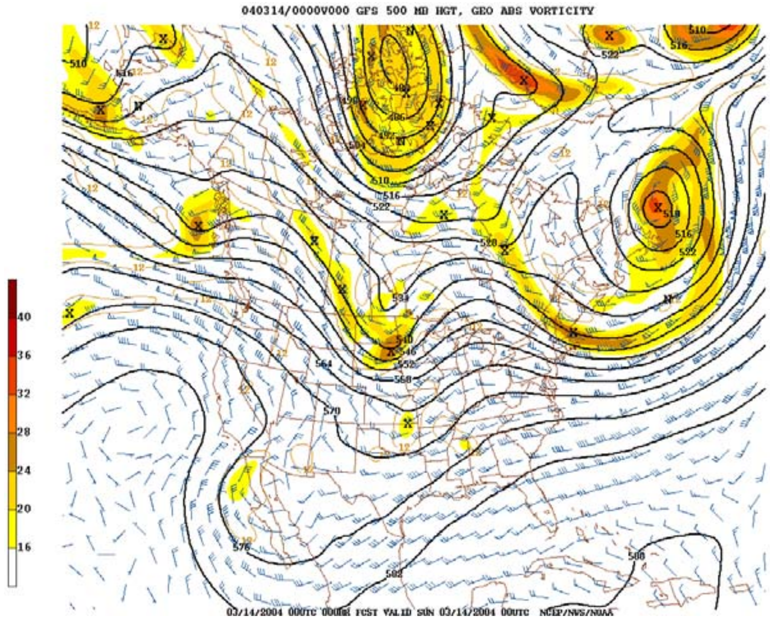
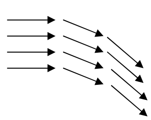


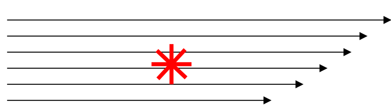
Vorticity

Vorticity is the microscopic measure of spin and rotation in a fluid.

Vorticity is defined as the curl of the velocity: $\nabla \times \vec{V}$



Wind direction varies → clockwise spin



Wind speed varies → clockwise spin

Absolute vorticity (inertial reference frame): $\vec{\omega}_a \equiv \nabla \times \vec{V}_a$

Relative vorticity (relative to rotating earth): $\vec{\omega} \equiv \nabla \times \vec{V}$

Expansion of relative vorticity into Cartesian components:

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\nabla \times \vec{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}$$

For large scale dynamics, the vertical component of vorticity is most important. The vertical components of absolute and relative vorticity in vector notation are:

$\zeta = \hat{k} \cdot \left(\nabla \times \vec{V} \right)$ relative vorticity

$\eta = \hat{k} \cdot \left(\nabla \times \vec{V}_a \right)$ absolute vorticity

From now on, vorticity implies the vertical component (unless otherwise stated.)

The absolute vorticity is equal to the relative vorticity plus the earth's vorticity. Since the earth's vorticity is

$$\hat{k} \cdot \left(\nabla \times \vec{V}_e \right) = 2\Omega \sin\phi = f$$

then

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{and} \quad \eta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f = \zeta + f$$

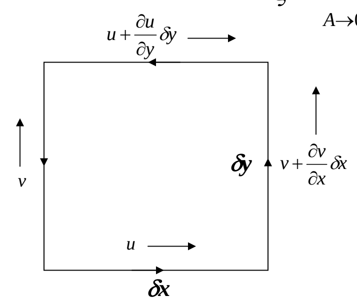
For large scale circulations, a typical magnitude for vorticity is

$$\zeta \approx \frac{U}{L} = 10^{-5}\ s^{-1}$$

Circulation and Vorticity

The relationship between relative vorticity ζ and circulation can be seen by considering the following expression, in which we will define the relative vorticity as the circulation about a closed contour in the horizontal plane divided by the area enclosed by that contour, in the limit as the area approaches zero.

$$\zeta = \lim_{A \to 0} \left(\oint \vec{V} \cdot d\vec{l} \right) A^{-1}$$



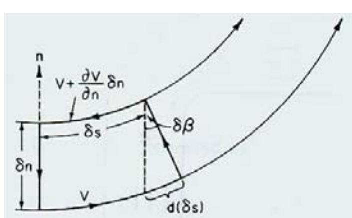
Evaluating $\vec{V} \cdot d\vec{l}$ for each side of the rectangle yields the circulation:

$$\delta C = u\,\delta x + \left(v + \frac{\partial v}{\partial x}\delta x \right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y \right)\delta x - v\,\delta y$$

$$\delta C = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\delta x \delta y \rightarrow \frac{\delta C}{\delta A} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \zeta$$

Vorticity in Natural Coordinates

Using natural coordinates can make it easier to physically interpret the relationship between relative vorticity and the flow. To express vorticity in natural coordinates, we compute the circulation around the infinitesimal contour shown below.



$$\delta C = V\left[\delta s + d(\delta s)\right] - \left(V + \frac{\partial V}{\partial n}\delta n \right)\delta s$$

From the diagram, $d(\delta s) = \delta \beta\, \delta n$, where $\delta\beta$ is the angular change in wind direction in the distance δn.

$$\delta C = \left(-\frac{\partial V}{\partial n} + V\frac{\partial \beta}{\partial s} \right)\delta n \delta s$$

In the limit $\delta n \delta s \rightarrow 0$,

$$\zeta = \lim_{\delta n, \delta s \to 0} \frac{\delta C}{\delta n \delta s} = \boxed{-\frac{\partial V}{\partial n} + \frac{V}{R_s}}$$

Vorticity in natural coordinates:

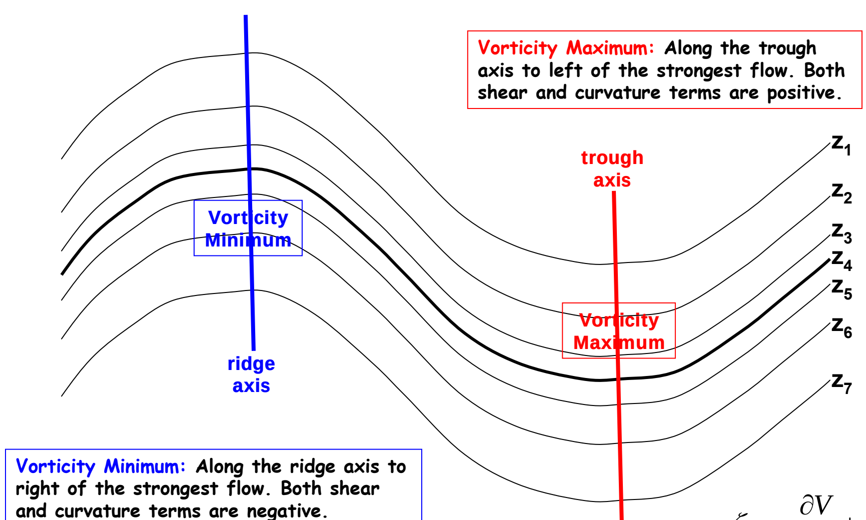
$$\boxed{\zeta = -\frac{\partial V}{\partial n} + \frac{V}{R_s}}$$

The relative vorticity is the sum of two parts:

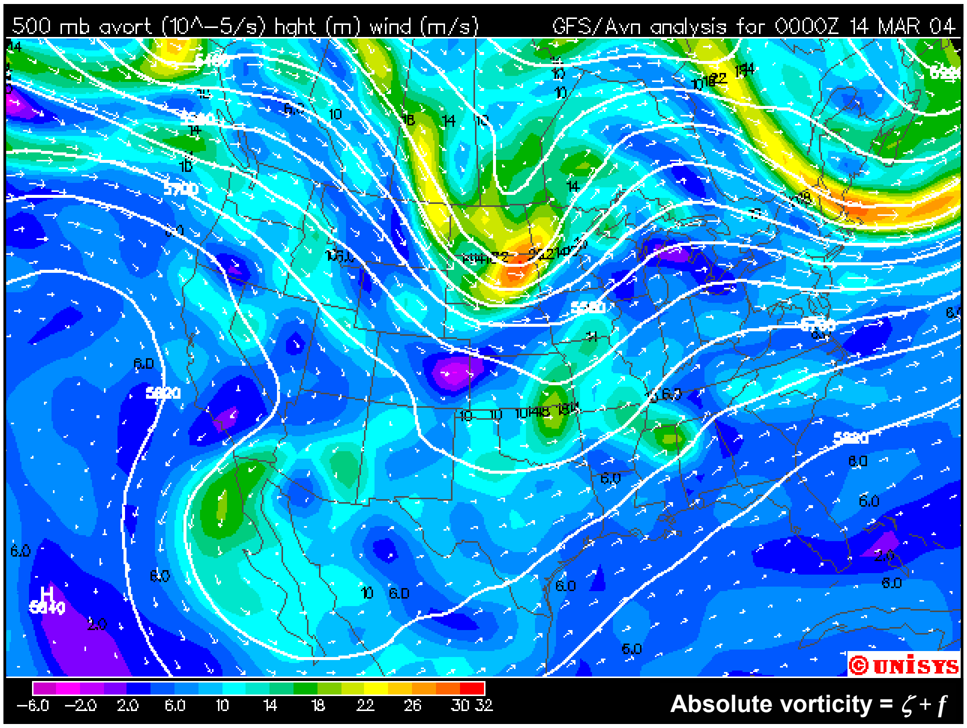
$-\dfrac{\partial V}{\partial n}$ The rate of change of wind speed normal to the direction of flow, which is called the shear vorticity.

$\dfrac{V}{R_s}$ The turning of the wind along a streamline, which is called the curvature vorticity.

Vorticity On The Weather Map



$$\zeta = -\frac{\partial V}{\partial n} + \frac{V}{R_s}$$



Potential Vorticity

Adiabatic flow can be described by Kelvin's circulation theorem:

$$\frac{d}{dt}\left(C + 2\Omega\,\delta A \sin\phi\right) = 0$$

where C is evaluated for a closed loop encompassing the area δA on an isentropic surface.

The vertical component of vorticity is given by $\zeta = \lim\limits_{A\to 0} \dfrac{C}{\delta A}$,

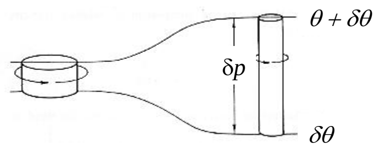
thus if the isentropic surface is approximately horizontal, for an infinitesimal parcel of air:

$$\frac{d}{dt}\left(\delta A\left(\zeta_\theta + f\right)\right) = 0 \rightarrow \delta A\left(\zeta_\theta + f\right) = const$$

relative vorticity on an isentropic surface

Coriolis parameter

Assume that this parcel is confined between potential temperature surfaces θ_0 and $\theta_0 + \delta\theta$, which are separated by $-\delta p$.



The mass of the parcel $\delta M = (-\delta p/g)\delta A$ must be conserved following the motion. Thus

$$\delta A = -\frac{\delta M g}{\delta p} = \left(-\frac{\delta\theta}{\delta p}\right)\left(\frac{\delta M g}{\delta\theta}\right) = const \times g\left(-\frac{\delta\theta}{\delta p}\right)$$

Substituting for δA in the previous expression and taking the limit $\delta p \rightarrow 0$ yields

Ertel's potential vorticity

$$\boxed{P \equiv \left(\zeta_\theta + f\right)\left(-g\frac{\partial\theta}{\partial p}\right) = const}$$

Units:
K kg^{-1} m^2 s^{-1}

Potential vorticity is conserved following adiabatic, frictionless flow.

General form of Ertel's potential vorticity: $P \equiv \left(\zeta_\theta + f\right)\left(-g\frac{\partial\theta}{\partial p}\right) = const$

Potential vorticity can be written in an even simpler form for a homogeneous, incompressible fluid. Since density is a constant, the horizontal area is inversely proportional to the depth of the fluid parcel.

$$\delta A = \frac{M}{\rho\,\delta z} = \frac{const}{\delta z}$$

For a homogeneous, incompressible fluid,

$$P = \frac{\left(\zeta + f\right)}{\delta z} = const$$

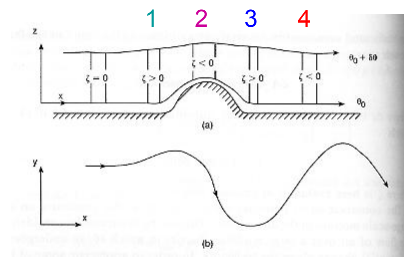
In the homogeneous, incompressible case as well as the general case, potential vorticity is a measure of the ratio of the absolute vorticity to the effective depth of the vortex. For the general case, the effective depth is the distance between potential temperature surfaces in pressure units ($-\partial\theta/\partial p$).

Westerly Flow Over a Barrier

Consider a westerly flow of air encountering a north-south mountain barrier. Upstream of the barrier, assume the flow is zonal and uniform, thus $\zeta = 0$.

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right) = const$$

A potential temperature surface near the ground approximately follows the terrain.



Pressure forces caused by the interaction of the flow with the mountain barrier cause the vertical displacement of an upper-level isentropic surface to be more spread out horizontally (i.e., greater width and smaller amplitude).

Region 1: Column stretches; $\zeta > 0$.
Region 2: Increase of latitude and reduction in column depth; $\zeta < 0$.
Region 3: Decrease of latitude and increase in column depth; $\zeta > 0$.
Region 4: Northward displacement and decrease in column depth; $\zeta < 0$.

The cyclonic flow downstream of a mountain barrier is known as a lee-side mountain trough, or lee trough.