Conservation of Mass

We consider a very small volume element of air

\[ \delta V = \delta x \delta y \delta z \]

that is centered at the point \((x_0, y_0, z_0)\).

Conservation of mass requires that the local rate of change of mass must equal the net rate of mass inflow per unit volume.

Rate of inflow of mass through left wall: \(M_{Lx}\)

Rate of outflow of mass through right wall: \(M_{Rx}\)

Net inflow of mass into the volume (x-direction only): \(M_x = M_{Lx} - M_{Rx}\)

Method: Use Taylor expansion to develop mathematical expressions for the inflow of mass through the walls of this fluid element.
Taylor series expansion:
\[ f(x) = f(x_0) + f'(x_0)(x - x_0) + \text{higher order terms} \]

In this case (neglecting higher order terms):
\[ \rho u(x) = \rho u(x_0) + \left. \frac{\partial}{\partial x} (\rho u)(x - x_0) \right|_{x=x_0} \]

Therefore, we can express the rates of inflow and outflow per unit area as:
\[ M_{Lx} = \left( \rho u - \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \right) \delta y \delta z \]
\[ M_{Rx} = \left( \rho u + \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \right) \delta y \delta z \]

Since the area of the left and right faces of the volume is \( \delta y \delta z \),
\[ \left[ \rho u - \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \right] \delta y \delta z - \left[ \rho u + \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \right] \delta y \delta z = -\frac{\partial}{\partial x} (\rho u) \delta x \delta y \delta z \]
is the net rate of mass flow into the volume due to the x velocity component.

We can derive similar expressions for the net rate of mass flow due to the y and z velocity components. Summing over all three components yields the net rate of mass inflow:
\[ -\left[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \right] \delta x \delta y \delta z \]
The net rate of mass inflow per unit volume must equal the rate of mass increase per unit volume, which is the local rate of change of density.

$$-\nabla \cdot (\rho \vec{V}) = \frac{\partial \rho}{\partial t}$$

**Mass divergence form of the continuity equation**

To derive an alternative form of the continuity equation, start with the mass divergence form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

Now apply the vector identity:

$$\nabla \cdot (\rho \vec{V}) = \rho \nabla \cdot \vec{V} + \vec{V} \cdot \nabla \rho$$

Next apply Euler's relationship:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{V} \cdot \nabla$$

$$\frac{1}{\rho} \frac{d\rho}{dt} + \nabla \cdot \vec{V} = 0$$

**Velocity divergence form of the continuity equation**
Scale Analysis of Continuity Equation

When scaling the continuity equation, it is important to recognize that large variations in density are only relevant in the vertical. So we represent the density field as the sum of a horizontal reference value and a deviation from that value:

$$\rho(x, y, z, t) = \rho_0(z) + \rho'(x, y, z, t)$$

Substitute into velocity divergence form of the continuity equation:

$$\frac{1}{\rho} \frac{d\rho}{dt} + \nabla \cdot \vec{V} = 0$$

$$\frac{1}{\rho_0} \left[ \frac{\partial}{\partial t} (\rho_0 + \rho') + \vec{V} \cdot \nabla (\rho_0 + \rho') \right] + \nabla \cdot \vec{V} = 0$$

If only the first three terms in the above equation were present, then the atmosphere would be incompressible.

The above result implies that the atmosphere behaves as if it were incompressible for purely horizontal motions (i.e., when \(w = 0\)).

\[ \frac{\rho' U}{\rho_0 L} = 10^{-7} \text{ s}^{-1} \]

\[ \frac{W}{H} = 10^{-6} \text{ s}^{-1} \]

\[ 10^{-3} \frac{U}{L'} \frac{W}{H} = 10^{-6} \text{ s}^{-1} \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + w \frac{d}{dz} \left( \ln \rho_0 \right) = 0 \]
Incompressible Fluids

\[ \rho + \nabla \cdot \vec{V} = 0 \]

If a fluid is incompressible, its density is constant following the fluid motion. Thus the velocity divergence must be zero for an incompressible fluid.

In the atmosphere, the compressibility associated with the height dependence of density is important when there is vertical motion.

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + w \frac{d}{dz} (\ln \rho_0) = 0 \]

Incompressibility is a very good approximation in the ocean, even when vertical motions are present.