Balanced Flow

• The pressure and velocity distributions in atmospheric systems are related by relatively simple, approximate force balances.
• We can gain a qualitative understanding by considering steady-state conditions, in which the fluid flow does not vary with time, and by assuming there are no vertical motions.
• To explore these balanced flow conditions, it is useful to define a new coordinate system, known as natural coordinates.

Natural Coordinates

• Natural coordinates are defined by a set of mutually orthogonal unit vectors whose orientation depends on the direction of the flow.

Unit vector $\hat{i}$ points along the direction of the flow.
Unit vector $\hat{n}$ is perpendicular to the flow, with positive to the left.
Unit vector $\hat{k}$ points upward.
Horizontal velocity: \( \vec{V} = \vec{V}_t \)

\( \vec{V} \) is the horizontal speed, which is a nonnegative scalar defined by \( V = ds/dt \), where \( s(x,y,t) \) is the curve followed by a fluid parcel moving in the horizontal plane.

To determine acceleration following the fluid motion,

\[
\frac{d\vec{V}}{dt} = \frac{d}{dt} \left( \vec{V}_t \right)
\]

\[
\frac{d\vec{V}}{dt} = \dot{r} \frac{dV}{dt} + V \frac{d\dot{r}}{dt}
\]

\[
\delta \psi = \frac{\delta s}{R} = \frac{\delta \vec{r}}{\vec{r}} = \delta \hat{t}
\]

\( R \) = radius of curvature (positive in positive n direction)

\( R > 0 \) if air parcels turn toward left
\( R < 0 \) if air parcels turn toward right

\[
\frac{d\dot{r}}{ds} = \frac{\dot{n}}{R} \quad (\text{taking limit as } \delta s \to 0)
\]

\[
\frac{d\dot{r}}{dt} = \frac{d\dot{r}}{ds} \frac{ds}{dt} = \frac{\dot{n}}{R} V
\]
\[
\frac{d\vec{V}}{dt} = i \frac{dV}{dt} + V \frac{d\hat{i}}{dt} \\
\frac{d\vec{V}}{dt} = i \frac{dV}{dt} + \hat{n} \frac{V^2}{R}
\]

vector form of acceleration following fluid motion in natural coordinates

\[- f\hat{k} \times \vec{V} = - fV\hat{n} \]

Coriolis (always acts normal to flow)

\[- \nabla_p \Phi = - \left( i \frac{\partial \Phi}{\partial s} + \hat{n} \frac{\partial \Phi}{\partial n} \right) \]

pressure gradient

\[
\frac{dV}{dt} = - \frac{\partial \Phi}{\partial s} \\
\frac{V^2}{R} + fV = - \frac{\partial \Phi}{\partial n}
\]

component equations of horizontal momentum equation (isobaric) in natural coordinate system

Balance of forces parallel to flow.

Balance of forces normal to flow.

For motion parallel to geopotential height contours, \(\frac{\partial \Phi}{\partial s} = 0\), which means that the speed is constant following the motion.

If the geopotential gradient normal to the direction of motion is constant along a trajectory, the normal component equation implies that the radius of curvature \(R\) is also constant.

When these assumptions are met we can define several simple categories of balanced flow that depend on the relative contributions of the three terms in the normal component equation.
Geostrophic Flow

Straight-line flow parallel to the height contours \( (R \to \pm \infty) \).

\[
\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}
\]

Horizontal components of Coriolis force and pressure gradient force are in exact balance.

\[
fV_s = -\frac{\partial \Phi}{\partial n}
\]

Inertial Flow

If the geopotential field is uniform on a constant pressure surface, then

\[
\frac{V^2}{R} + fV = -\frac{\partial \Phi^0}{\partial n}
\]

\[
R = -\frac{V}{f}
\]

Because uniform geopotential implies constant speed, then the radius of curvature is constant if we assume \( f \) is approximately constant. Air parcels will follow circular paths in anticyclonic rotation with period

\[
P = \left| \frac{2\pi R}{V} \right| = \frac{2\pi}{f}
\]
Cyclostrophic Flow

If the horizontal scale of an atmospheric disturbance is sufficiently small, such as in a tornado, waterspout, or dust devil, then the Coriolis term will be substantially smaller than the pressure gradient and centrifugal terms:

\[
\frac{V^2}{R} + f^2 = -\frac{\partial \Phi}{\partial n}
\]

\[
V = \left( -R \frac{\partial \Phi}{\partial n} \right)^{1/2}
\]

There are four possible orientations for the direction of curvature and the pressure gradient:

- If \( R > 0 \) and \( \frac{\partial \Phi}{\partial n} < 0 \), then \( V \) is real.
- If \( R < 0 \) and \( \frac{\partial \Phi}{\partial n} < 0 \), then \( V \) is imaginary.
- If \( R > 0 \) and \( \frac{\partial \Phi}{\partial n} < 0 \), then \( V \) is imaginary.
- If \( R < 0 \) and \( \frac{\partial \Phi}{\partial n} > 0 \), then \( V \) is real.

Only low pressure systems can have cyclostrophic flow. Cyclostrophic flow can either be clockwise or counterclockwise.
Cyclostrophic Flow in the Real World

- Because the Coriolis acceleration is neglected in cyclostrophic flow, there should be no preferred direction of rotation.
- In fact, waterspouts and (especially) dust devils can show both clockwise and counterclockwise rotation.
- Most (but not all) tornadoes in the Northern Hemisphere rotate counterclockwise, because they develop from large, rotating supercell thunderstorms. Because of their relatively large scale (~10 km), supercell thunderstorms do feel the effects of Coriolis acceleration.

Gradient Flow

For large-scale weather disturbances in which the flow is not required to be in a straight line, a three-way balance among the Coriolis, centrifugal, and pressure gradient forces exists.

\[
\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}
\]

\[
V = -\frac{fR}{2} \pm \left( \frac{f^2 R^2}{4} - R \frac{\partial \Phi}{\partial n} \right)^{1/2}
\]

There are four possible orientations for the direction of curvature and the pressure gradient, each with two roots:

<table>
<thead>
<tr>
<th></th>
<th>$R &gt; 0$</th>
<th>$R &lt; 0$</th>
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</thead>
<tbody>
<tr>
<td>$\partial \Phi / \partial n &gt; 0$</td>
<td>Positive root: unphysical</td>
<td>Positive root: unphysical</td>
</tr>
<tr>
<td></td>
<td>Positive root: anomalous low</td>
<td>Positive root: anomalous high</td>
</tr>
<tr>
<td></td>
<td>Negative root: unphysical</td>
<td>Negative root: regular high</td>
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</table>
Geostrophic Wind vs. Gradient Wind

\[ \frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n} \]

\[ \frac{V^2}{R} + fV - fV_g = 0 \]

\[ \frac{V_g}{V} = 1 + \frac{V}{fR} \]

Start with balance of forces normal to flow.

Rewrite using definition of geostrophic wind.

Solve for ratio of geostrophic wind to gradient wind.

For normal cyclonic flow \((R > 0)\) the geostrophic wind overestimates the gradient wind, while for anticyclonic flow \((R < 0)\) the geostrophic wind underestimates the gradient wind.

Rossby number

The geostrophic wind is a good estimate of the gradient wind when the Rossby number is small.